Foundations of Blockchains
Lecture #3: Simulation, Indistinguishability, and the Necessity of PKI
(ROUGH DRAFT)*

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1 The Upshot

1. Impossibility results are important because they educate you about why you can't have everything you want and the fundamental trade-offs that any solution must make.

2. Impossibility results in computer science are hard to prove because the design space is so rich.

3. Impossibility result of this lecture: when at least one third of the nodes can be Byzantine (i.e., $f \geq n/3$), there is no protocol for the Byzantine broadcast (BB) problem (defined last lecture) that satisfies termination, agreement, and validity.

4. This impossibility result works in the permissioned and synchronous setting but without the PKI assumption. As such, it does not contradict the guarantees proved for the Dolev-Strong protocol in Lecture 2.

5. Thus, for the design of distributed protocols, cryptographic and trusted setup assumptions fundamentally change the game and enable solutions that otherwise could not exist.

6. The rough intuition of the proof (for the three-node case) is that an honest node may have enough information to deduce that one of the other two nodes is Byzantine but (because either could be framing the other) not enough to deduce which one it is.

†Email: tim.roughgarden@gmail.com. The preparation of these notes was partially supported by a seed grant from the Columbia-IBM Center for Blockchain and Data Transparency. This lecture's expository strategy is heavily influenced by the Decentralized Thoughts blog (https://decentralizedthoughts.github.io/).

7. The two key themes in the proof are simulation (as a generic way for a Byzantine node to send conflicting messages) and indistinguishability (a catch-22 faced by an honest node that faces two plausible scenarios, each demanding a different action).

8. The key step of the proof is to set up a "hexagon thought experiment" in which six machines with inconsistent initialization files run copies of an allegedly correct protocol.

9. This hexagon effectively encodes three different three-node BB instances. Correctness of the assumed protocol constrains the outputs of the honest nodes in these instances.

10. A Byzantine node can force the two honest nodes to operate identically to how they would in the hexagon experiment by simulating the remaining four nodes in the hexagon. Constraints on node outputs in the BB instances then transfer to the hexagon experiment.

11. The three constraints implied by the BB instances are mutually inconsistent in the hexagon experiment, completing the contradiction and showing that the assumed correct protocol cannot exist.

12. The proof breaks down if the PKI assumption is imposed because a Byzantine node is then unable to forge the signatures necessary to carry out a faithful simulation of the four hexagon nodes.

2 On Impossibility Results

Recap and context. Last lecture, we proved a possibility result showing that, under a list of assumptions, there is a state machine replication (SMR) protocol satisfying the two properties that we care about, consistency and liveness. We did this by first reducing the multi-shot SMR problem to the single-shot Byzantine broadcast problem (i.e., showing how to build a good protocol for the former from one for the latter), and then presenting the Dolev-Strong protocol, which guarantees termination, agreement, and validity the Byzantine broadcast problem with any number of Byzantine nodes.

Beginning with this lecture and continuing in Lectures 4–6, we'll switch our focus from possibility results to impossibility results (before returning to a possibility result for the Tendermint protocol in Lecture 7). That is, we'll identify sets of assumptions under which there are no protocols that satisfy all the properties that we want. This lecture revisits the Byzantine broadcast problem from Lecture 2 and proves the "FLM impossibility result," which states that, under assumptions incomparable to the ones we made for the Dolev-Strong protocol, there is no protocol that satisfies all of termination, agreement, and validity. There are several reasons to spend some quality time with this impossibility result:

1. It's part of the distributed computing canon, and has a super-cool proof.

2. It highlights the importance of cryptographic and trusted setup assumptions (like the PKI assumption from Lecture 2).

3. It introduces two recurring themes in proofs of impossibility results in distributed computing: the idea of an adversary who performs a simulation of one or more honest nodes, and the idea of indistinguishability (in which an honest node gets caught in a catch-22 and can't distinguish between different worlds in which different behavior would be expected).

Digression: The point of impossibility results. Theory is great for proving possibility results, as we saw last lecture with the Dolev-Strong protocol. But arguably, theory might be even more uniquely suited for *impossibility* results that delineate what computers, algorithms, and protocols *cannot* do.

Thinking back on the history of computer science, perhaps the first academic computer science paper ever was Alan Turing's 1936 paper that introduced the Turing machine model of computation [5].[1] The main result of that paper was an impossibility result showing that computers will never be able to solve the halting problem. Thus impossibility results have been part of the fabric of computer science literally from day zero as an academic field. A somewhat more modern example would be the development of NP-completeness, which explains why certain computational problems appear unsolvable by efficient algorithms. Distributed computing is another part of computer science that is defined in part by its deep and illuminating impossibility results. A lot of the richness of distributed computing as an academic discipline lies in the subtle curvature of the frontier between what can and cannot be done (as a function of exactly which assumptions you make).

Impossibility results seems depressing. What are they good for? The point of an impossibility result is definitely *not* for some academic in an ivory tower to scold someone that they can't or shouldn't attempt to solve some problem. No matter what theorems I prove in these lectures, people are not going to keep building new blockchain protocols and putting them out into the world (as they should). Rather, an impossibility result teaches you why you can't always have everything that you want, and about the compromises that are going to be required when you tackle a problem.

For example, if you compare the major blockchain consensus protocols (sometimes called "layer-1s"), you'll see that none are perfect, and each has its own disappointing weaknesses. You might then wonder: "Why shouldn't I, or some other super-smart person, design one protocol to rule them all, with all the good points and none of the bad points of other protocols?" An impossibility result can inform us that, alas, no matter how smart any of us might be, such a protocol does not exist. It also provides you with a lens through which to evaluate and compare different blockchain protocols and the different compromises they (inevitably) make—for example, learning to identify which protocols "favor safely over liveness" or vice versa (more on this in later lectures).

Finally, an impossibility result can also indicate that you're working in too demanding a model, with too few assumptions. For example, the FLP impossibility result (covered in Lectures 4–5) rules out good consensus protocols in the extremely demanding "asynchronous

[1]Note that this is nearly a decade before any actual computers (in the sense meant by that word today) existed! Chalk another one up for the power of theory and mathematical abstraction.

model," and this result will guide us toward the definition of the "partially synchronous model" in Lecture 6. (Whereas without the FLP impossibility result, it's not clear anyone would have bothered to write down that model.) As we'll see, the partially synchronous model will be the perfect sweet spot between more extreme models—its assumptions are weak enough that it forces you to design protocols that really are useful in the real world, yet strong enough to escape the FLP impossibility result and allow for provable guarantees.

3 The FLM Impossibility Result

3.1 The Byzantine Broadcast Problem

Recall from Lecture 2 the definition of and goals for the Byzantine broadcast problem (and see that lecture for further discussion). There is an a set $\{1, 2, 3, \ldots, n\}$ of nodes—this is common knowledge to all when a protocol commences, as is the identity of a *sender* node. The sender has a *private input* $v^* \in V$. (For this lecture's impossibility result, we'll only need to use set $V = \{0, 1\}$.)

The goal is to design a protocol with three properties:

Desired Properties of a Byzantine Broadcast Protocol

1. **Termination.** Every honest node i eventually halts with some output $v_i \in V$.

2. **Agreement.** All honest nodes halt with the same output (whether or not the sender is honest).

3. **Validity.** *If the sender is an honest node,* then the common output of the honest nodes is the private input v^* of the sender.

3.2 Statement of the Impossibility Result

The impossibility result in this lecture was first established by Pease, Shostak, and Lamport [3] (the same authors from the "Byzantine generals" paper mentioned last lecture, with their names in a different order). We'll present a later (super-slick) proof by Fischer, Lynch, and Merritt [1]. For the theorem statement, recall that we use f to denote a known upper bound on the maximum number of Byzantine nodes, where a Byzantine node can deviate from the intended protocol in arbitrary ways. (The bigger the f, the harder it is for honest nodes to not get confused and achieve consensus.)

Theorem 3.1 ([3, 1]) *In the synchronous model with $f \geq \frac{n}{3}$, there is no Byzantine broadcast protocol that satisfies termination, agreement, and validity.*

This is, Byzantine broadcast is possible in the synchronous model only if more than two-thirds of the nodes are honest and correctly follow the protocol. Here "synchronous model" is the same model of communication that we used for the Dolev-Strong protocol in the last lecture—all nodes share a global clock, and every message sent in a time step arrives at its recipient prior to the beginning of the next time step.

3.3 Cryptography and Trusted Setups Matter!

Looking at the statement of Theorem 3.1, you should have a question: Didn't we just (in Lecture 2) learn a protocol (the Dolev-Strong protocol) that *does* solve the Byzantine broadcast problem (in the synchronous model) *no matter how many* Byzantine nodes there are? Yes we did! But neither the proof from last lecture nor the forthcoming proof of Theorem 3.1 are incorrect. The reason there's no contradiction is that the two results apply under slightly different sets of assumptions (in particular, different cryptographic/trusted setup assumptions). So, as we go through the proof of Theorem 3.1 in the next section, your homework is keep an eye out for any steps in the proof that wouldn't apply to the Dolev-Strong protocol (and there must be some, since that protocol escapes the impossibility result). We'll discuss this point at length in Section 5.

3.4 Two Simplifying Assumptions

To keep the length of this lecture reasonable, let's make two simplifying assumptions (both of which can be removed with a bit more work). First, let's focus on deterministic protocols (with no random coin flips). Allowing randomization doesn't really save you from the impossibility result (see [2]), but let's not worry about it further. Second, let's focus on the simplest-possible case of the impossibility result, with $n = 3$ and $f = 1$ (three nodes, one of which might be Byzantine). Your first reaction might be that this assumption trivializes the result, but the truth is the exact opposite—this special case already captures all of the complexity and nuance of the general impossibility result. A good homework problem is to extend this lecture's argument for the special case of $n = 3$ and $f = 1$ to a proof of Theorem 3.1 in its full glory. One approach is to use a reduction—that is, to show how to use a protocol that allegedly solves the Byzantine broadcast problem for some n and $f \geq n/3$ to build a different protocol that solves the $n = 3$ and $f = 1$ case (which would be a contradiction). Alternatively, the proof in the next section can be reworked to directly apply to your favorite choice of n and $f \geq n/3$.

3.5 Some Vague Intuition

Before getting to the formal proof of Theorem 3.1, let me give you some intuition about why the result might be true, and what's fundamentally driving the magical threshold of $n/3$. (By the way, if you skip the proof, don't forget to read Section 5 for a discussion of why the Dolev-Strong protocol doesn't contradict Theorem 3.1.) Warning: the intuition will be somewhat vague. The super-slick proof in Section 4 more or less encodes this intuition, though there is some distance between them.

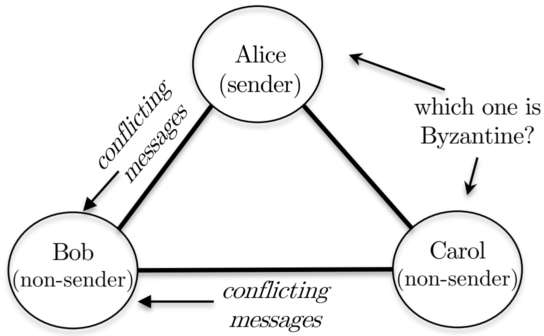


Figure 1: Vague intuition for Theorem 3.1. Bob can't tell if Alice (the sender) is Byzantine and Carol is honestly echoing her messages or if Carol is Byzantine and trying to frame Alice.

Think of a set of three nodes, call them Alice, Bob, and Carol, shown as vertices in Figure 1. The edges in the figure indicate that each of three nodes knows about and can communicate with the other two nodes. Assume that Alice is the designated sender.

The basic idea is that one node may have enough information to deduce that one of the other two nodes is Byzantine but not enough to deduce which one it is. For example, assume the role of Bob. He might well hear conflicting information from Alice (the sender) and Carol (the other non-sender), for example if Carol is supposed to repeat to Bob any messages she received from Alice. One plausible scenario is that Carol is honestly echoing the messages she heard from Alice, but Alice is Byzantine and sending conflicting messages to Bob. But it's also totally possible that Alice is honest and sent out consistent information, but Carol is Byzantine and attempting to frame Alice. Unable to determine who is to blame, Bob doesn't know what he should output. (Whereas if $n = 4$ and $f = 1$, Bob could plausibly figure out the right output by resorting to some type of majority vote over the other three nodes.)

If you're really on top of things, you might be suspicious of this informal argument (in particular, of Carol's ability to frame Alice) and have an inkling about why an argument of this type might not apply to the Dolev-Strong protocol. In any case, spot the inevitable step(s) of the proof in the next section that doesn't apply to the Dolev-Strong protocol!

4 The FLM Proof of Theorem 3.1

4.1 Preamble (Optional)

The FLM proof of Theorem 3.1 is impressively clever—almost like they already knew the result was true and wanted to back out the slickest proof possible. (Actually, maybe that's exactly what they were doing—recall that the theorem was proved earlier in [3].) No one who sees this proof thinks it would be easy to come up with, and it's normal to not grok it the first time you see it. On a first reading, your main goal should be convince yourself that it's correct. With some repeated readings and rumination, you'll be able to internalize it.

The challenge of impossibility results. The main reason that impossibility results in computer science are so challenging is the richness of the design space—you need to rule out any possible solution, no matter how creative or crazy. This is exactly what made Turing's impossibility result [5] such a breakthrough—no algorithm, present or future, no matter how wild, could ever solve the halting problem. For another example, you might have heard about the P $\neq$ NP conjecture, which roughly asserts that there are no algorithms for any NP-hard problems (satisfiability, traveling salesman problem, etc.) that are guaranteed to improve substantially over (exponential-time) exhaustive search (see e.g. [4] for details). And the whole reason nobody has been able to prove the conjecture yet is the richness of the algorithm design space. With so many crazy possibilities for polynomial-time algorithms—ever seen Strassen's matrix multiplication algorithm?)—how could one ever rule all of them out?

The exact same issue can make impossibility results for distributed protocols difficult. For example, suppose we try to prove Theorem 3.1 by contradiction. That is, we assume that the theorem is false and that there in fact a protocol (call it π) for the Byzantine broadcast problem with $n = 3$ and $f = 1$ guaranteed to satisfy termination, agreement, and validity. If we can derive a contradiction (showing that our initial assumption is false), we'll have completed the proof. Here's the thing, though: we have no idea what π looks like, as it could be arbitrarily crazy event-driven piece of code. How can we write down a single argument that somehow works simultaneously for all of the infinitely possible π's?

Simulation and indistinguishability. There is one thing we can do in the proof that uses the assumed protocol π in a generic way: run it! That is, we (or an adversary) can imagine deploying it on one or more nodes and then seeing what happens. (Proofs in theoretical computer science often use algorithms in this "black-box" way. E.g., reread any NP-completeness proof that you might have seen in the past.) This idea of "simulation"—in our case, of hypothetical honest nodes by a devious Byzantine node—is one of the two key themes to look for in the FLM proof of Theorem 3.1.

The second theme of the proof is "indistinguishability," which we alluded to in the vague intuition given in Section 3.5. Indistinguishability refers to a catch-22 situation faced by an honest node, in which the node does not have enough information to figure out which of two plausible scenarios is the actual reality. If validity and/or agreement demand different outputs in the two cases, then the node is hopelessly stuck (whatever it outputs, it will be an incorrect output for one of the plausible scenarios).

With a specific protocol, we can start imagine ways in which a Byzantine node might trap an honest node into a catch-22 situation (along the lines of Section 3.5). But how can we do it in a generic way, that applies no matter what π is?

4.2 The Hexagon (a Thought Experiment)

Next is the really, really ingenuous step of the Fischer-Lynch-Merritt proof, which involves a thought experiment carried out on a 6-cycle of nodes (a "hexagon"). (Once you've convinced yourself that this thought experiment makes sense, the rest of the proof won't be too bad.)

The basic idea is to effectively make two copies of each of Alice, Bob, and Carol, in order to force them to participate simultaneously in multiple worlds that demand different things of them. Accordingly, we're going to buy six machines and deploy an allegedly correct Byzantine broadcast protocol π on each of them.

To make sense of this idea, let's be clear on the inputs that the protocol π is expecting to find when it's first fired up on some node i (in a locally stored initialization file, if you like):

(I1) the names and IP addresses of two other nodes (recall the protocol π is designed for the case of $n = 3$);

(I2) among node i and the two other nodes, which one is the sender (there should be exactly one sender);

(I3) if node i is the sender, its private input.

The designer of the protocol π surely had in mind a scenario in which the initialization files at i's two neighbors (call them j and k) list the nodes i, k and i, j, respectively—this would be the case in any bona fide instance of Byzantine broadcast with $n = 3$, and these are the only cases that π's designer is responsible for. *However*, at least in principle, we could run the protocol π at node i even if j's and k's initialization files did not satisfy this consistency property. We can't count on π satisfying any properties in this case; for example, the initialization file inconsistencies might well cause the protocol to run forever. But because π is just a piece of code and node i's initialization file has all the requisite ingredients, we can nonetheless press play on node i and see what happens.

The hexagon thought experiment exploits the idea above, running six copies of π on machines with inconsistent initialization files (Figure 2, with edges indicating pairs of nodes that know about each other before the protocol starts):

Thought Experiment Instructions

1. Buy a node A and set it up with the following initialization file: (i) A's neighbors are the nodes B and C (specified by name and IP address); (ii) A is a sender while B, C are non-senders; (iii) A's private input is 0.

2. Buy a node B and set it up with: (i) B's neighbors are the nodes A and C'; (ii) A is a sender while B, C' are non-senders. (Think of C' as having the same name as C (e.g., each thinks it's the real node #7) but a different IP address.)

3. Buy and setup a node C with: (i) neighbors A and B'; (ii) A is a sender

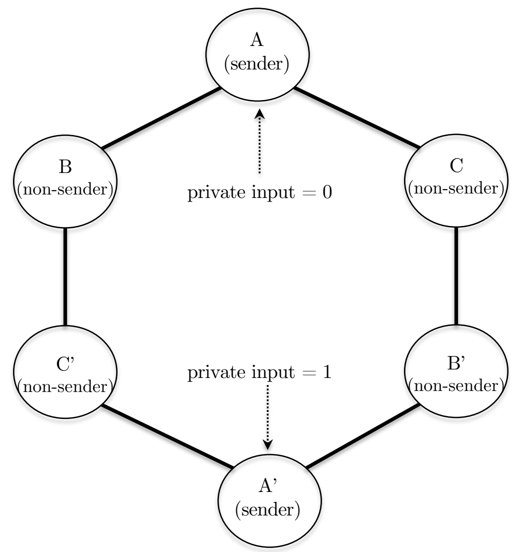


Figure 2: The hexagon thought experiment. Each machine runs the protocol π with the indicated two neighbors (and, if a sender, with the indicated private input).

while B', C are non-senders.

4. Buy and setup a node C' with: (i) neighbors A' and B; (ii) A' is a sender while B, C' are non-senders.

5. Buy and setup a node B' with: (i) neighbors A' and C; (ii) A' is a sender while B', C are non-senders.

6. Buy and setup a node A' with: (i) neighbors B' and C'; (ii) A' is a sender while B', C' are non-senders; (iii) the private input of A' is 1.

This whole endeavor should seem bizarre. The only point to understand now is that, given the code of a protocol π, no one can stop you from buying six machines, installing π on each of them, and setting up their initialization files as above. Because each initialization file has exactly the information (I1)–(I3) expected by π—for all the node knows, it's participating in a bona fide instance of Byzantine broadcast with $n = 3$—no one can stop you from pressing play simultaneously on all six machines. All bets are off about what happens next (π was not designed with this experiment in mind), but no one can stop you from running the experiment to see what the nodes' eventual outputs will be.

4.3 Proof Strategy

Because the initialization file of each node in the hexagon thought experiment typechecks with the expectations of the protocol π, you can run the experiment of running π simultaneously on all six nodes. But the overall setting in the thought experiment ($n = 6$, not all nodes no about all other nodes) certainly doesn't typecheck with the one π is designed for ($n = 3$, set of nodes common knowledge). Thus any properties that π might have in the latter scenario (e.g., validity) need not carry over to the former scenario (e.g., with two senders with different private inputs, it's not even clear what validity should mean). This point has important consequences for our proof strategy.[2]

In our proof, to derive a contradiction, presumably we are going to use the assumptions that π satisfies termination, agreement, and validity. (Achieving any two of the three properties is easy, so there's no way to prove an impossibility result without using all three assumptions.) How are we going to use the assumed properties of π in the proof, given that we can't appeal to them directly in the hexagon thought experiment?

To build a bridge between the hexagon experiment (where π has no guarantees) and bona fide three-node instances of Byzantine broadcast (where π satisfies termination, agreement, and validity), we're going to show that the hexagon effectively encodes three different three-node Byzantine broadcast instances. In each of these instances, agreement or validity will impose a constraint on the outputs of a pair of nodes, and these constraints will carry over to the hexagon. We'll see that the three constraints are mutually incompatible, contradicting the fact that the hexagon thought experiment must have some well-defined output.

4.4 Byzantine Broadcast Instance #1

Let's see the first instance of Byzantine broadcast that is effectively "embedded"in the hexagon through experiment. Consider the instance shown in Figure **??**(a), with a Byzantine sender (node X) and two honest non-senders (B and C'). As a bona fide instance of Byzantine broadcast (with $n = 3$ and one Byzantine node), we can appeal to the promised guarantees of the assumed protocol π, specifically its termination and agreement properties (validity is irrelevant because the sender is Byzantine). Remember what these properties say: *no matter what* a Byzantine node does, no matter how crazy its strategy, the honest nodes must eventually terminate, and upon termination output the same value.

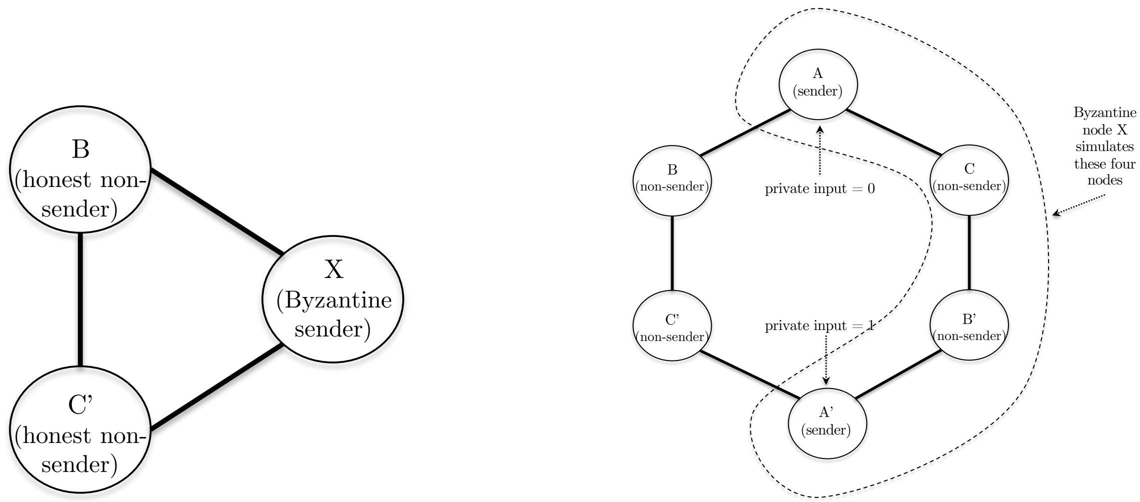
Intuitively, the protocol π must be robust to X sending out conflicting messages. But given that we know nothing about how π works, how can we know which messages might confused the nodes that are running π honestly? Could there be a generic "send conflicting messages" strategy for X that applies to every possible protocol π?

Here's one crazy strategy that the Byzantine node X in Figure 3 could do: *simulate* the behavior of the four nodes

$$A \leftrightarrow C \leftrightarrow B' \leftrightarrow A'$$

in the hexagon thought experiment, in which A', C, B' and A' run π honestly with the initialization parameters specified in Section 4.2. (Remember, simulation is one of the two key themes of this proof.) Think of it this way: the Byzantine node X spins up four virtual machines on its node, each running a separate copy of π with the appropriate initialization parameters. The Byzantine node monitors the progress of each copy of π. Whenever π

[2] If the bird's-eye view in this section is too vague for you, revisit it after absorbing the details in Sections 4.4–4.6.



(a) Byzantine broadcast instance (b) Hexagon thought experiment

Figure 3: Left: Byzantine broadcast instance #1 (with $n = 3$ and one Byzantine node). Right: by simulating four nodes in the hexagon experiment, the Byzantine node X can force B and C' to operate identically in the triangle and in the hexagon. The agreement property of π dictates that B and C' output the same value.

instructs the virtual machine masquerading as A to send a message to A's neighbor B (which is X's actual neighbor in the Byzantine broadcast instance), node X sends that message to B (over the communication network). Similarly, any messages X receives from B (over the communication network) are forwarded to the virtual machine on X that is running π as A. Messages between the virtual machine running as A' and the (actual) node C' are handled analogously, over the communication network. Finally, whenever one virtual machine wants to send a message to another one (e.g., messages between the virtual machines running as C and B'), the Byzantine node X can directly deliver that message to the destination machine, without ever touching the communication network. In the end, the Byzantine node X interacts with its neighbor B as if it were the node A in the hexagon (running π honestly with private input 0) and with C' as if it were the node A' in the hexagon (running π honestly with private input 1). This strategy is well defined for any protocol π and thus serves as our generic way for a Byzantine node to send conflicting messages.

At this point you should agree that this "simulate four nodes on the hexagon" strategy is something that the Byzantine node X *could* do. But what's the point of this crazy strategy? The point is that it tricks the honest nodes B and C' to behave exactly *as if* they were in the hexagon experiment. That is, because the initialization parameters and the sequences of messages received by B and C' are exactly the same in the triangle in Figure 3(a) (by construction of X's strategy) and in the hexagon in Figure 3(b), they are kept in the dark cannot distinguish which is the actual reality and must operate identically in both cases. (Remember, indistinguishability is the other key theme of the proof.)

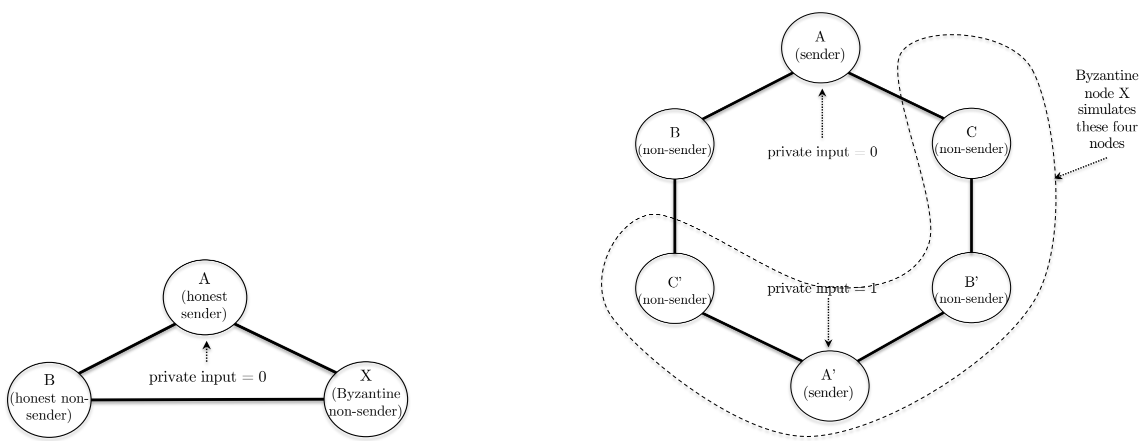
Now we can see how to translate π's assumed properties in bona fide Byzantine broadcast instances (like the triangle in Figure 3(a), with the above Byzantine node strategy) to properties of its behavior in the hexagon experiment. Specifically, in the triangle, because π satisfies termination and agreement, nodes B and C' eventually output something, and the outputs must be the same. (This statement is true no matter what strategy X uses.) Because B and C' operate identically in the triangle (with the specified strategy for X) and in the hexagon thought experiment, the outcome must be the same:

in the hexagon thought experiment, the nodes B and C' output the same value. (1)

Good news: If you followed the argument in this section, then you've completed all of the hard parts of understanding the FLM proof of Theorem 3.1. We need to talk through two more scenarios before arriving at a contradiction, but the pattern of those two arguments will be the same as this one.

4.5 Byzantine Broadcast Instance #2

The first scenario relied on the assumed agreement of the protocol π; the second and third ones will rely on validity. Validity applies only with an honest sender, so in these two scenarios the Byzantine node X will be a non-sender (see the triangle in Figure 4(a)).



(a) Byzantine broadcast instance

(b) Hexagon thought experiment

Figure 4: Left: Byzantine broadcast instance #2. Right: by simulating four nodes in the hexagon experiment, the Byzantine node X can force A and B to operate identically in the triangle and in the hexagon. The validity property of π dictates that A and B both output 0.

The honest nodes A and B in the triangle play the same roles as their namesakes in the hexagon, and particular the honest sender A has a private input of 0. To keep the nodes A and B in the dark as to whether they reside in the triangle or the hexagon, the Byzantine non-sender X in the triangle runs four copies of π (in separate virtual machines) to simulate the remaining four nodes of the hexagon (Figure 4(b)):

$$C \leftrightarrow B' \leftrightarrow A' \leftrightarrow C'.$$

That is, the Byzantine node X interacts with the honest sender A as if it were the node C in the hexagon, and with the honest non-sender B as if it were the node C' in the hexagon (while also simulating the nodes B' and A' to keep track of which messages C and C' would be sending).

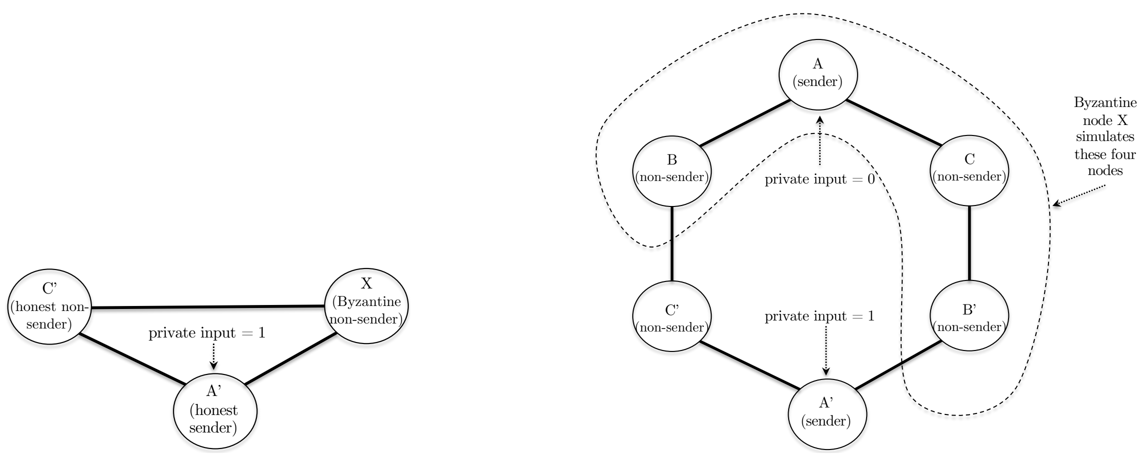
Because π satisfies termination and validity in every bona fide three-node instance of Byzantine broadcast (like the triangle) with an arbitrary Byzantine node strategy (like X's simulation strategy), nodes A and B must eventually output 0 (i.e., A's private input) on the triangle. By virtue of operating identically in the hexagon experiment (by construction of X's strategy), those two nodes must also eventually output 0 in the hexagon:

$$\textit{in the hexagon thought experiment, the nodes } A \textit{ and } B \textit{ output 0.} \tag{2}$$

Again, the seemingly bizarre simulation strategy by the Byzantine node X is what enables the transfer of constraints imposed on nodes' outputs in the triangle to those in the hexagon.

4.6 Byzantine Broadcast Instance #3

The third scenario is very similar to the second one, with an honest sender A' (this time with private input 1), an honest non-sender C', and a Byzantine non-sender X (see the triangle in Figure 4(a)).



(a) Byzantine broadcast instance (b) Hexagon thought experiment

Figure 5: Left: Byzantine broadcast instance #3. Right: by simulating four nodes in the hexagon experiment, the Byzantine node X can force A' and C' to operate identically in the triangle and in the hexagon. The validity property of π dictates that A' and C' both output 1.

To keep the honest nodes A' and C' in the dark as to whether they reside in the triangle or the hexagon, the Byzantine non-sender X runs four copies of π to simulate the remaining four nodes of the hexagon (Figure 4(b))

$$B \leftrightarrow A \leftrightarrow C \leftrightarrow B',$$

interacting with A' as if it were B' and with C' as if it were B.

Because π satisfies termination and validity in every bona fide three-node instance of Byzantine broadcast (like the triangle) with an arbitrary Byzantine node strategy (like X's simulation strategy), nodes A' and C' must eventually output 1 (i.e., the private input of the honest sender A') on the triangle. By virtue of operating identically in the hexagon experiment (by construction of X's strategy), those two nodes must also eventually output 1 in the hexagon:

in the hexagon thought experiment, the nodes A' and C' output 1. (3)

4.7 Completing the Contradiction

We looked at three different three-node Byzantine broadcast instances and (appealing to π's assumed guarantees) deduced constraints on the outputs of the honest nodes in each. We chose a simulation strategy for the Byzantine node in each of these instances to force the honest nodes to operate identically to how they would act in the hypothetical hexagon thought experiment defined in Section 4.2, thereby transferring the constraints from the three instances over to hexagon. These constraints (1)–(3) assert that, in the hexagon thought experiment, nodes B and C' must terminate and output the same thing; B must output 0; and C' must output 1. Whatever the outcome of the hexagon experiment might be (and it must be something), it can't possibly satisfy these three mutually inconsistent constraints. This completes the contradiction, implying that the assumed protocol π cannot exist. That is, there cannot be a Byzantine broadcast protocol with $n = 3$ and $f = 1$ that guarantees termination, agreement, and validity.[3]

5 Discussion

The main result of Lecture 2 is that, in the synchronous model, you can solve the Byzantine broadcast problem (i.e., with a protocol that satisfies termination, agreement, and validity) with any number of Byzantine nodes. The main result of this lecture is that, in the synchronous model, you *can't* solve the Byzantine broadcast problem if at least one-third of the nodes can be Byzantine. What gives?

5.1 Resolving the Contradiction

No prizes for guessing the answer, which appears in the title of this lecture. In Lecture 2, we introduced the public key infrastructure (PKI) assumption as an extension of the usual permission assumption: not only do all the nodes know the names and IP addresses of all the nodes, but also their public keys. (Each node is assumed to have a distinct public key-private

[3]Mystified? Totally normal. On a first read, your goal should simply be to convince yourself that the proof is legitimate. If you really want to internalize the proof and the fundamental drivers of the impossibility result, budget some time to run back over it in your mind a number of times.

key pair, with the private key known only to the node and the public key known to all.) This is an example of a *trusted setup* assumption, asserting that a certain computation (in this case, public key distribution) is done correctly in advance of the protocol's commencement, remaining silent on how this computation might actually happen.

Our analysis of the Dolev-Strong protocol in Lecture 2 relied on three assumptions: the permissioned setting (with all node names common knowledge at the start of the protocol); synchronous setting (reliable message delivery); and the PKI assumption (so that nodes begin the protocol with the ability to verify each others' signatures). The proof in Section 4 does not violate the permissioned setting assumption—in all three of the Byzantine broadcast instances considered, the node set is known to all nodes up front. Neither does it violate the synchronous setting assumption—it relies only on devious communication strategies for Byzantine nodes and not on any manipulation of the timing of message deliveries. Thus, by the process of elimination, we can conclude that Theorem 3.1 cannot possibly remain true under the PKI assumption (if it did, it would contradict what we've already proved about the Dolev-Strong protocol). Given that the theorem is false with the PKI assumption, the proof in Section 4 must somehow violate that assumption. But now exactly would common knowledge of nodes' public keys break the proof?

5.2 Can We Salvage the Proof?

For the hexagon thought experiment in Section 4.2 to even make sense under the PKI assumption, we need to add the relevant cryptographic keys to nodes' initialization files. The new file format is:

(I1) the names, IP addresses, and public keys of two other nodes;

(I2) among node i and the two other nodes, which one is the sender;

(I3) if node i is the sender, its private input;

(I4) a public key-private key pair for node i (distinct from those of the other two nodes).

The most obvious way to modify the thought experiment would then be (see Figure 6):

Revised Thought Experiment

1. Setup node A with: (i) neighbors B and C (specified by name, IP address, and public key); (ii) A is a sender while B, C are non-senders; (iii) A's private input is 0 (iv) a public key-private key pair (pk_A, sk_A).

2. Setup node B with: (i) neighbors A and C'; (ii) A is a sender while B, C' are non-senders; (iv) a public key-private key pair (pk_B, sk_B). (Think of C' as having the same name and key pair as C (e.g., each thinks it's the real node #7 and has the corresponding credentials) but a different IP address.)

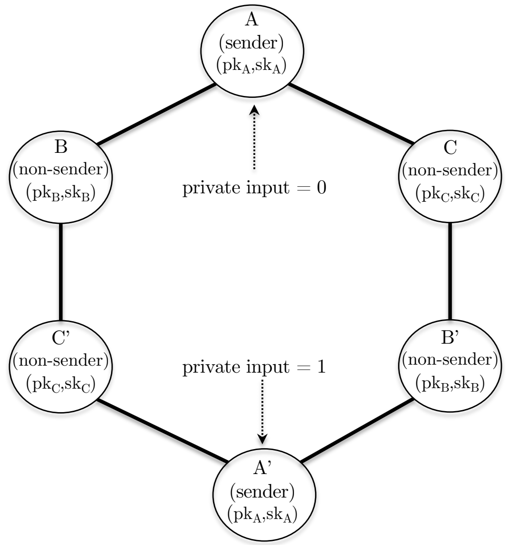
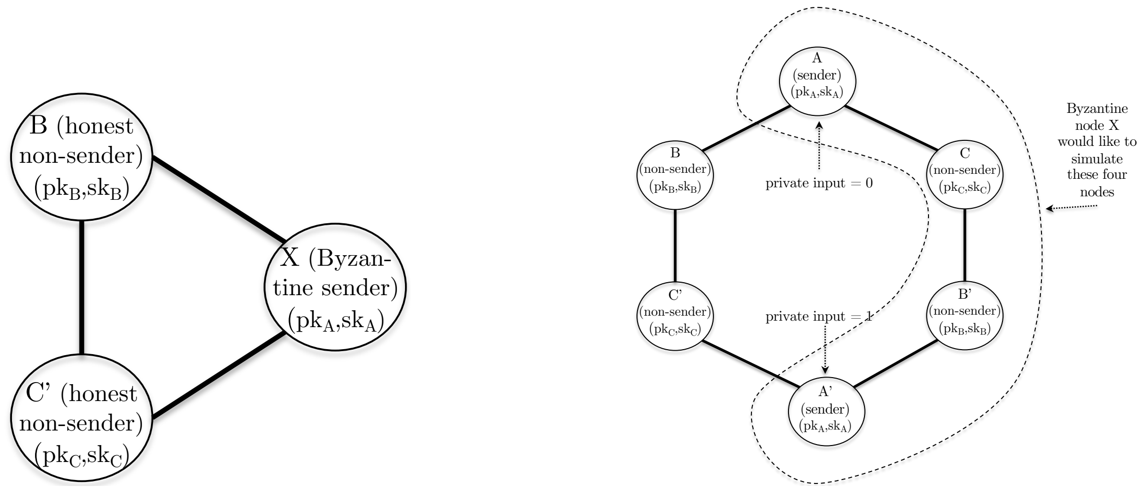


Figure 6: Extension of the hexagon thought experiment to incorporate the PKI assumption.

3. Setup node C with: (i) neighbors A and B'; (ii) A is a sender while B', C are non-senders; (iv) a public key-private key pair (pk_C, sk_C).
4. Setup node C' with: (i) neighbors A' and B; (ii) A' is a sender while B, C' are non-senders; (iv) a public key-private key pair (pk_C, sk_C).
5. Setup node B' with: (i) neighbors A' and C; (ii) A' is a sender while B', C are non-senders; (iv) a public key-private key pair (pk_B, sk_B).
6. Setup node A' with: (i) neighbors B' and C'; (ii) A' is a sender while B', C' are non-senders; (iii) the private input of A' is 1; (iv) a public key-private key pair (pk_A, sk_A).

Now that we have a well-defined thought experiment with the PKI assumption, let's try to replicate the argument in Section 4. (The proof hasn't broken yet, but it has to break somewhere...) The next step is to consider a Byzantine broadcast instance analogous to the one in Section 4.4 (Figure 7(a)) and the strategy for the Byzantine node X in which it simulates the other four vertices in the revised hexagon experiment (Figure 7(b)). Remember that the point of this strategy is to keep the two honest nodes in the dark as to whether they reside in the triangle or the hexagon, in which case any constraints we can deduce for nodes' behavior in the triangle would carry over to the hexagon.

But is the Byzantine node X really in a position to pull off such a simulation? The assumed Byzantine broadcast protocol π might well instruct every node to sign every message it sends (as in the Dolev-Strong protocol). The adversary X knows its public key-private key pair (pk_A, sk_A) in the Byzantine broadcast instance, as well as the other two public keys pk_B and pk_C, but it *does not know* the other two private keys sk_B and sk_C. Thus, while the adversary is perfectly positioned to simulate the nodes A and A'—the only ones the honest



(a) Byzantine broadcast instance

(b) Hexagon thought experiment

Figure 7: An attempt to extend the argument of Figure 3 to the PKI setting. The simulation strategy could require the forging of signatures without knowledge of the appropriate private key, and so the Byzantine node X cannot carry it out.

nodes B and C' communication with directly—it *cannot* sign a message on behalf of nodes B' or C and is therefore *unable to simulate them*. (Remember our permanent assumptions that cryptography exists and that a signature for an as-yet-unseen message cannot be forged without knowledge of the appropriate private key.) In sum: our proof of Theorem 3.1 made crucial use of strategies in which a Byzantine node simulates multiple other honest nodes, and these strategies become impossible to carry out under the PKI assumption.

5.3 Cryptography and Trusted Setups Matter!

The fact that there are things you can do with the PKI assumption (the main result of last lecture) that you provably cannot do without it (the main result of this lecture) is super-interesting. We now understand that assumptions about the existence of secure digital signature schemes and public key distribution fundamentally affects what you can and cannot do with a consensus protocol.

For a point of contrast, think about algorithms. If you're trying to design a faster algorithm for the minimum spanning tree problem, who cares what kind of cryptography exists? Or suppose you in the middle of tackling an NP-hard problem like the traveling salesman problem, and I hand you on a silver platter a secure digital signature scheme (or some more fancy cryptographic primitive). You would stare at me quizzically: "What am I supposed to do this this?" Whereas, for the design of distributed protocols, cryptographic assumptions fundamentally change the game and enable solutions that otherwise would not exist.

Way back in Section 2 we mentioned that one purpose (among many) of impossibility results is to indicate when you're working in too demanding a model, with too few assumptions. This lecture's main result is a textbook example—it shows that if you want a consensus protocol that is robust to an arbitrary number of faulty nodes, you have no choice but to get your hands on a secure digital signature scheme and figure out how to get everyone's public keys to everyone else (or pull off some other trusted setup assumption that makes simulation strategies infeasible for Byzantine nodes). The impossibility result in the next two lectures (in which we drop the assumption of the synchronous setting) is another textbook example—though rather than educating us about the need for trusted setup assumptions, it will guide us toward necessary assumptions on the reliability of the underlying communication network.

References

[1] M. J. Fischer, N. A. Lynch, and M. Merritt. Easy impossibility proofs for distributed consensus problems. *Distributed Computing*, 1(1):26–39, 1986. URL: `https://groups.csail.mit.edu/tds/papers/Lynch/FischerLynchMerritt-dc.pdf`.

[2] R. L. Graham and A. C. Yao. On the improbability of reaching Byzantine agreements. In *Proceedings of the 21st Annual ACM Symposium on Theory of Computing (STOC)*, pages 467–478, 1989. URL: `https://mathweb.ucsd.edu/~ronspubs/89_08_byzantine.pdf`.

[3] M. Pease, R. Shostak, and L. Lamport. Reaching agreement in the presence of faults. *Journal of the ACM*, 27(2):228–234, 1980. URL: `https://lamport.azurewebsites.net/pubs/reaching.pdf`.

[4] T. Roughgarden. *Algorithms Illuminated, Part 4: Algorithms for NP-Hard Problems*. Soundlikeyourself Publishing, 2020.

[5] A. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society, Series 2*, 42:230–265, 1936. Erratum: 43:544–546, 1937. URL: `https://www.astro.puc.cl/~rparra/tools/PAPERS/turing_1936.pdf`.